ON THE GROWTH OF GRADED POLYNOMIAL IDENTITIES OF sl_n

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ABSTRACT. Let $\mathbb{K}$ be a field of characteristic 0 and L be a G-graded Lie PI-algebra, where G is a finite group. We define the graded Gelfand-Kirillov dimension of L. Then we measure the growth of the $\mathbb{Z}_n$-graded polynomial identities of the Lie algebra of $n \times n$ traceless matrices $sl_n(\mathbb{K})$ giving an exact value of its $\mathbb{Z}_n$-graded Gelfand-Kirillov dimension.

1. INTRODUCTION

All fields we refer to are to be considered of characteristic 0. Let A be an algebra with non-trivial polynomial identity (or simply PI-algebra) and denote by $T(A)$ the T-ideal of its polynomial identities. In general the description of a T-ideal is a hard problem. In order to overcome this difficulty one introduces some functions measuring the growth of $T(A)$ in some sense. For every integer $k \geq 1$ one defines the Gelfand-Kirillov (GK) dimension of A in k variables as the Gelfand-Kirillov dimension of the relatively free algebra of rank k with respect to the ideal of polynomial identities of A. The Gelfand-Kirillov dimension of arbitrary finitely generated algebra is a measure of the rate of growth in terms of any finite generating set. For any associative PI-algebra the GK dimension of A in k variables is always an integer (see [4] and [5]). For a more detailed background about Gelfand-Kirillov dimension see the book of Krause and Lenagan [18].

After the powerful theory developed by Kemer in the 80's in order to solve the *Specht problem*, i.e., the existence of a finite generating set for any T-ideal over a field of characteristic zero, other kinds of identities have become object of much interest in the theory of polynomial identities. For example if A is a G-graded algebra, where G is a group, one may be interested in the study of G-graded polynomial identities of A. In [1] Aljadeff and Kanel-Belov proved the analog of the Specht problem in the graded case, for G-graded PI-algebra when G is a finite group. We have to cite the work by Sviridova [25] who solved the Specht problem for associative graded algebras graded by a finite abelian group.

When the group G is finite, Centrone in [9] defined the G-graded GK dimension in k graded variables for a G-graded PI-algebra. In particular if A is a G-graded PI algebra, one may consider the relatively-free G-graded algebra of A in k variables $\mathcal{F}_k^G(A)$ and define the G-graded Gelfand-Kirillov dimension of A as the GK dimension of $\mathcal{F}_k^G(A)$. In [10] Centrone computed the graded Gelfand-Kirillov dimension

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of the *verbally prime* algebras $M_n(\mathbb{K})$, $M_n(E)$ and $M_{a,b}(E)$, where $\mathbb{K}$ is a field and E is the infinite dimensional Grassmann algebra, endowed with a "Vasilovsky type"-grading (see [26]). We recall that the verbally prime algebras are the building blocks of the theory of Kemer.

The graded polynomial identities of Lie algebras have seldom been studied. Most of the known results about the topic are related to the algebra of 2×2 traceless matrices sl_2. Razmyslov found a finite basis of the ordinary identities satisfied by sl_2 and proved that the variety of Lie algebras generated by sl_2 is Spechtian i.e., the identities of any subvariety have a finite basis ([23]). Up to graded isomorphism, sl_2 can only be graded by $\{0\}$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}$. The structure of the relatively-free algebra for each non-trivial grading over a field of characteristic 0 was described by Repin in [24]. In [17] Koshlukov described the graded polynomial identities of sl_2 for the above gradings when the base field is infinite and of characteristic $\neq 2$. Recently Giambruno and Souza proved in [15] that the variety of graded Lie algebras generated by sl_2 is also Spechtian.

In this paper we define the graded Gelfand-Kirillov dimension on a finite set of graded variables of any G-graded Lie PI-algebra according to the associative case. We compute explicitly the graded Gelfan-Kirillov dimension of sl_2 over a field of characteristic 0 for each of its non-trivial gradings. Next we consider the Lie algebra sl_n of $n \times n$ traceless matrices endowed with the grading of Vasilovsky, then we measure the growth of the graded polynomial identities of the Lie algebra of $n \times n$ traceless matrices sl_n giving the exact value of their graded Gelfand-Kirillov dimension. We recall that the ordinary GK dimension in k variables of the relatively free algebra of sl_2 has been already computed with two different methods in [14] and [19].

2. Preliminaries

Throughout the paper $\mathbb{K}$ will be a fixed field of characteristic zero, L a Lie algebra over $\mathbb{K}$ and $sl_n = sl_n(\mathbb{K})$ the Lie algebra of $n \times n$ traceless matrices over $\mathbb{K}$. Moreover we refer to a *left-normed* product of elements $l_1, \ldots, l_k$ of a Lie algebra as the product $[[\cdots[[l_1, l_2], l_3]\cdots], l_k]$.

Recall that the growth function $g_V(n)(A)$ of a finitely generated algebra A generated by the set $V = \{v_1, \ldots, v_r\}$ is the dimension of the space spanned by words of length at most n. We say that $g_V(n)(A)$ is the growth function with respect to the vector space generated by V over $\mathbb{K}$. The Gelfand-Kirillov dimension of A is the superior limit

$$\mathrm{GKdim}(A) = \limsup_{n \to +\infty} \frac{\ln g_V(n)(A)}{\ln n}$$

when it exists. The Gelfand-Kirillov dimension does not depend on the choice of the generators of algebra, then we shall use $g(n)(A)$ instead of $g_V(n)(A)$.

From now on G will be an abelian group and L be a G-graded Lie algebra over $\mathbb{K}$. Recall that L is a G-graded Lie algebra if $L = \bigoplus_{g \in G} L_g$ is a direct sum of subspaces such that $L_g L_h \subseteq L_{g+h}$, for all $g, h \in G$.

The free G-graded Lie algebra $\mathcal{L}(X)$ is the G-graded Lie algebra freely generated on the set $X = \bigcup_{g \in G} X_g$, where for any $g \in G$ the sets $X_g = \{x_i^g; i \geq 1\}$ of variables of homogeneous degree g are infinite and disjoint. A polynomial f of $\mathcal{L}(X)$ is a G-graded polynomial identity of L if f vanishes under all graded substitutions i.e.,

for any $g \in G$, we evaluate the variables x_i^g into elements of the homogeneous component L_g. We denote by $T_G(L)$ the ideal of $\mathcal{L}(X)$ of G-graded polynomial identities of L. It is easily checked that $T_G(L)$ is a T_G-ideal i.e., an ideal invariant under all G-graded endomorphisms of $\mathcal{L}(X)$. We say that L is a graded PI-algebra if $T_G(L) \neq 0$.

Here we shall always consider gradings on L such that the support $\mathrm{Supp}(L) = \{g \in G : L_g \neq 0\}$ is finite. Suppose for instance that $\mathrm{Supp}(L) = \{g_1, \dots, g_s\}$. We denote by

$$\mathcal{L}_k^G(L) = \frac{\mathcal{L}\left(x_1^{g_1}, \dots, x_k^{g_1}, \dots, x_1^{g_s}, \dots, x_k^{g_s}\right)}{\mathcal{L}\left(x_1^{g_1}, \dots, x_k^{g_1}, \dots, x_1^{g_s}, \dots, x_k^{g_s}\right) \cap T_G(L)}$$

the relatively-free G-graded algebra of L in k variables. We define the G-graded Gelfand-Kirillov dimension of L in k variables similarly to the associative case (see [9, 10]),

$$\mathrm{GKdim}_k^G(L) := \mathrm{GKdim}(\mathcal{L}_k^G(L)).$$

See the survey of Drensky [12] and Centrone [8] for more details on the Gelfand-Kirillov dimension of PI-algebras.

Suppose that L is a G-graded PI-algebra. It is well known that the relatively-free G-graded algebra $\mathcal{L}_k^G(L)$ in the variables $\bar{x}_1^{g_1}, \dots, \bar{x}_k^{g_1}, \bar{x}_1^{g_2}, \dots, \bar{x}_k^{g_2}, \dots, \bar{x}_1^{g_s}, \dots, \bar{x}_k^{g_s}$, where $\bar{x}_i^g$ is the homomorphic image of x_i^g in $\mathcal{L}_k^G(L)$, is a $\mathbb{Z}^\infty$-graded Lie algebra when considering the usual multigrading. The *Hilbert series* of $\mathcal{L}_k^G(L)$ in the variables $t_1, t_2, \dots, t_r$, $r = sk$ is the formal power series

$$H(L, t_1, t_2, \dots, t_s) = \sum_{m=(m_1, m_2, \dots, m_s)} \dim \mathcal{L}_k^G(L)^{(m_1, m_2, \dots, m_s)} t_1^{m_1} t_2^{m_2} \cdots t_s^{m_s},$$

where $\mathcal{L}_k^G(L)^{(m_1, m_2, \dots, m_s)}$ is the homogeneous component of degree $(m_1, m_2, \dots, m_s)$.

The growth function of $\mathcal{L}_k^G(L)$ with respect to the vector space generated by

$$V = \{\bar{x}_1^{g_1}, \dots, \bar{x}_k^{g_1}, \dots, \bar{x}_1^{g_s}, \dots, \bar{x}_k^{g_s}\}$$

is

$$g(n)(\mathcal{L}_k^G(sl_2)) = \sum_{m \leq n} a_m,$$

where

$$a_m = \sum_{m = m_1 + m_2 + \cdots + m_s} \dim_{\mathbb{K}} \mathcal{L}_k^G(L)^{(m_1, m_2, \dots, m_s)}. \tag{1}$$

Let T be a tableau of shape σ filled in with natural numbers $\{1, \dots, k\}$ and let d_i be the multiplicity of i in T. A tableau is said to be *semistandard* if the entries weakly increase along each row and strictly increase down each column. We say that

$$S_\sigma(t_1, \dots, t_k) = \sum_{T_\sigma \text{ semistandard}} t_1^{d_1} t_2^{d_2} \cdots t_k^{d_k}$$

is the *Schur Function of* σ *in the variables* $t_1, \dots, t_k$.

The G-graded cocharacters of L are strictly related with the Hilbert series $H(L, t_1, t_2, \dots, t_r)$. We have the following (see [7],[13]).

Proposition 2.1. *Let L be a G-graded Lie algebra and $m_1, \dots, m_s \geq 0$. Suppose that*

$$\chi_{m_1, \dots, m_s}(L) = \sum_{\langle\sigma\rangle \vdash m} m_{\langle\sigma\rangle} \chi_{\sigma_1} \otimes \cdots \otimes \chi_{\sigma_s}$$

is the $(m_1,\dots,m_s)$-cocharacter of L where $\langle\sigma\rangle=(\sigma_1,\dots,\sigma_s)$ is a multipartition of n i.e., $\sigma_1\vdash m_1,\dots,\sigma_s\vdash m_s$ and $m=m_1+\cdots+m_s$. Then

$$H(L,t_1,t_2,\dots,t_r)=\sum_{m=(m_1,m_2,\dots,m_r)} m_{\sigma_1,\dots,\sigma_s}S_{\sigma_1}(T_1)\cdots S_{\sigma_s}(T_s) \tag{2}$$

where $S_{\sigma_1}(T_1)$, …, $S_{\sigma_s}(T_s)$ are Schur functions with shape $\sigma_1,\dots,\sigma_s$ in the variables $T_1=\{t_1,t_2,\dots,t_k\}$, …, $T_s=\{t_{sk-k+1},t_{sk-k+2},\dots,t_r\}$.

3. Graded Gelfand-Kirilov dimension for sl_2

In this section we calculate the G-graded Gelfand-Kirillov dimension of sl_2, for any non-trivial group G.

In order to simplify the notation we write

$$h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\qquad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},$$

for the standard basis of $L=sl_2$. Up to equivalence, sl_2 has three non-trivial G-gradings (see [2]):

1. $G=\mathbb{Z}_2$: $L_0=\mathbb{K}h$, $L_1=\mathbb{K}e\oplus\mathbb{K}f$;
2. $G=\mathbb{Z}_2\times\mathbb{Z}_2$: $L_{(0,0)}=0$, $L_{(1,0)}=\mathbb{K}h$, $L_{(0,1)}=\mathbb{K}(e+f)$, $L_{(1,1)}=\mathbb{K}(e-f)$;
3. $G=\mathbb{Z}$: $L_{-1}=\mathbb{K}e$, $L_0=\mathbb{K}h$, $L_1=\mathbb{K}f$, $L_i=0,\ i\notin\{-1,0,1\}$.

We shall write sl_2^G for the algebra sl_2 endowed with the correspondent G-grading (when necessary). We consider each grading separately.

We recall that $\lfloor w\rfloor=\max\{u\in\mathbb{Z}\ :\ u\le w\}$ is the integer part of w when w is a real number.

3.1. The $\mathbb{Z}_2$-grading.

We assume that $L=sl_2$ is $\mathbb{Z}_2$-graded i.e., $L=L_0\oplus L_1$ where $L_0=\mathbb{K}h$ and $L_1=\mathbb{K}e\oplus\mathbb{K}f$. In order to compute the $\mathbb{Z}_2$-graded Gelfand-Kirillov dimension of sl_2 we use the Repin's description of the $\mathbb{Z}_2$-graded cocharacter sequence of sl_2.

Proposition 3.1 ([24]). *Let*

$$\chi_{p,n-p}(sl_2^{\mathbb{Z}_2})=\sum_{\sigma\vdash p,\,\tau\vdash n-p} m_{\sigma,\tau}\chi_\sigma\otimes\chi_\tau$$

be the $(p,n-p)$-th cocharacter of sl_2. Then, for every $\sigma\vdash p$ and $\tau\vdash n-p$, $m_{\sigma,\tau}\le 1$. Moreover, $m_{\sigma,\tau}=1$ if and only if $\sigma=(p)$, $\tau=(q+r,q)$ and the following conditions hold:

1. *$p\ne n$;*
2. *$r\ne n$;*
3. *$r\equiv 1$ or $p+q\equiv 1$(mod 2).*

Then we have the following result.

Proposition 3.2. *The growth function $g(n)=g(n)(\mathcal{L}_k^{\mathbb{Z}_2}(sl_2))$ of $\mathcal{L}_k^{\mathbb{Z}_2}(sl_2)$ is a polynomial in n of degree $3k-1$.*

Proof. By Proposition 3.1,

$$\chi_{p,m-p}(sl_2^{\mathbb{Z}_2})=\sum \boxed{p}\otimes\begin{array}{|l|}\hline m-p-q\\\hline q\\\hline\end{array}$$

where $p \neq m$, $r = m - p - 2q \neq m$ and $r \equiv 1$ or $p + q \equiv 1 (\text{mod } 2)$. If m is even we have that

$$\chi_{p,m-p}(sl_2^{\mathbb{Z}_2}) = \sum_{p_1=0}^{\frac{m-2}{2}} \sum_{q_1=0}^{\lfloor \frac{m-(2p_1+1)}{2} \rfloor} \boxed{2p_1+1} \otimes \begin{array}{|l}\hline m-(2p_1+1)-q_1 \\ \hline q_1 \\ \hline \end{array}$$

$$+ \sum_{p_2=0}^{\frac{m-2}{2}} \sum_{t_2=0}^{\lfloor \frac{m-2p_2-2}{4} \rfloor} \boxed{2p_2} \otimes \begin{array}{|l}\hline m-2p_2-(2t_2+1) \\ \hline 2t_2+1 \\ \hline \end{array}.$$

By the definition of Hilbert series given in (1) and (2) we get:

$$a_m = \sum_{p_1=0}^{\frac{m-2}{2}} \sum_{q_1=0}^{\lfloor \frac{m-(2p_1+1)}{2} \rfloor} s_T\left(\boxed{2p_1+1}\right) s_U\left(\begin{array}{|l}\hline m-(2p_1+1)-q_1 \\ \hline q_1 \\ \hline \end{array}\right)$$

$$+ \sum_{p_2=0}^{\frac{m-2}{2}} \sum_{t_2=0}^{\lfloor \frac{m-2p_2-2}{4} \rfloor} s_T\left(\boxed{2p_2}\right) s_U\left(\begin{array}{|l}\hline m-2p_2-(2t_2+1) \\ \hline 2t_2+1 \\ \hline \end{array}\right),$$

where $s_T\left(\boxed{p}\right)$, $s_U\left(\begin{array}{|l}\hline m-p-q \\ \hline q \\ \hline \end{array}\right)$ is the number of semistandard tableaux with shape $\boxed{p}$, $\begin{array}{|l}\hline m-p-q \\ \hline q \\ \hline \end{array}$ in the variables T and U respectively. For m odd we observe that a_m can be calculated as above. Since (see [16, 27])

$$\begin{aligned} s_T\left(\boxed{p}\right) &= \binom{p+k-1}{k-1} \\ s_U\left(\begin{array}{|l}\hline m-p-q \\ \hline q \\ \hline \end{array}\right) &= \frac{m-p-2q+1}{k-1}\binom{m-p-q+k-1}{k-2}\binom{q+k-2}{k-2} \end{aligned}$$

we can assume that for all m, a_m is asymptotically equivalent to

$$\sum_{p_1=0}^{\frac{m-2}{2}} \sum_{q_1=0}^{\lfloor \frac{m-(2p_1+1)}{2} \rfloor} s_T\left(\boxed{2p_1+1}\right) s_U\left(\begin{array}{|l}\hline m-(2p_1+1)-q_1 \\ \hline q_1 \\ \hline \end{array}\right) =$$

$$\sum_{p_1=0}^{\frac{m-2}{2}} \binom{2p_1+k}{k-1} \sum_{q_1=0}^{\lfloor \frac{m-(2p_1+1)}{2} \rfloor} \frac{m-2p_1-2q_1}{k-1} \binom{m-2p_1-2q_1-k-2}{k-2}\binom{q_1+k-2}{k-2}.$$

Using standard combinatorial arguments, it is easy to see that a_m is a polynomial in m of degree $3k-2$. Therefore $g(n)$ is a polynomial in n of degree $3k-1$ since $g(n) = \sum_{m \leq n} a_m$. $\square$

As an immediate consequence of the previous proposition we obtain the $\mathbb{Z}_2$-graded Gelfand-Kirillov dimension of sl_2.

Corollary 3.3. $\mathrm{GKdim}_k^{\mathbb{Z}_2}(sl_2) = 3k-1$.

3.2. **The $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.** We consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading of $L = sl_2$ where $L_{(0,0)} = 0$, $L_{(1,0)} = \mathbb{K}h$, $L_{(0,1)} = \mathbb{K}(e+f)$ and $L_{(1,1)} = \mathbb{K}(e-f)$.

The next result by Repin [24] shows the $\mathbb{Z}_2 \times \mathbb{Z}_2$-cocharacters decomposition of sl_2.

Proposition 3.4. *Let*

$$\chi_n(sl_2^{\mathbb{Z}_2\times\mathbb{Z}_2}) = \sum_{\substack{\sigma\vdash p,\,\tau\vdash q\\ \pi\vdash r}} m_{\sigma,\tau,\pi}\chi_\sigma\otimes\chi_\tau\otimes\chi_\pi$$

be the (p,q,r)-th cocharacter of sl_2. Then, for every $\sigma\vdash p$, $\tau\vdash q$, $\pi\vdash r$, $m_{\sigma,\tau,\pi}\leq 1$. Moreover, $m_{\sigma,\tau,\pi}=1$ if and only if $\sigma=(p),\tau=(q),\pi=(r)$ and the following conditions hold:

1. *$p\neq n$, $q\neq n$, $r\neq n$;*
2. *$p+q\equiv 1$ or $q+r\equiv 1$(mod 2).*

We have the following result.

Proposition 3.5. *The growth function $g(n)=g(n)(\mathcal{L}_k^{\mathbb{Z}_2\times\mathbb{Z}_2}(sl_2))$ of $\mathcal{L}_k^{\mathbb{Z}_2\times\mathbb{Z}_2}(sl_2)$ is a polynomial in n of degree $3k+1$.*

Proof. By Proposition 3.4,

$$\chi_{p,q,r}(sl_2^{\mathbb{Z}_2\times\mathbb{Z}_2}) = \sum \boxed{p}\otimes\boxed{q}\otimes\boxed{r}$$

where $m=p+q+r$, $p\neq m$, $q\neq m$, $r\neq m$ and $p+q\equiv 1$ or $q+r\equiv 1$(mod 2).

If m is even then

$$\begin{aligned}\chi_{p,q,r}(sl_2^{\mathbb{Z}_2\times\mathbb{Z}_2}) = & \sum_{s_1=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{t_1=0}^{\lfloor\frac{m-2s_1-1}{2}\rfloor}\sum_{u_1=0}^{\frac{m-2s_1-2t_1-2}{2}} \boxed{2s_1}\otimes\boxed{2t_1+1}\otimes\boxed{2u_1+1}\\
&+\sum_{s_2=0}^{\frac{m-2}{2}}\sum_{t_2=0}^{\frac{m-2s_2-2}{2}}\sum_{u_2=0}^{\frac{m-2s_2-2t_2-2}{2}} \boxed{2s_2+1}\otimes\boxed{2t_2+1}\otimes\boxed{2u_2}\\
&+\sum_{s_3=0}^{\frac{m-2}{2}}\sum_{t_3=0}^{\lfloor\frac{m-2s_3-1}{2}\rfloor}\sum_{u_3=0}^{\frac{m-2s_3-2t_3-2}{2}} \boxed{2s_3+1}\otimes\boxed{2t_3}\otimes\boxed{2u_3+1}.\end{aligned}$$

Using arguments similar to those of Proposition 3.2 we have that

$$\begin{aligned}a_m &\approx \sum_{s_1=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{t_1=0}^{\lfloor\frac{m-2s_1-1}{2}\rfloor}\sum_{u_1=0}^{\frac{m-2s_1-2t_1-2}{2}} s_T\left(\boxed{2s_1}\right)s_U\left(\boxed{2t_1+1}\right)s_V\left(\boxed{2u_1+1}\right)\\
&= \sum_{s_1=0}^{\lfloor\frac{m-1}{2}\rfloor}\sum_{t_1=0}^{\lfloor\frac{m-2s_1-1}{2}\rfloor}\sum_{u_1=0}^{\frac{m-2s_1-2t_1-2}{2}} \binom{2s_1+k-1}{k-1}\binom{2t_1+k}{k-1}\binom{2u_1+k}{k-1}\end{aligned}$$

for all m. It is easy to see that a_m is a polynomial in m of degree $3k$. Therefore $g(n)$ is a polynomial in n of degree $3k+1$. □

Corollary 3.6. $\mathrm{GKdim}_k^{\mathbb{Z}_2\times\mathbb{Z}_2}(sl_2)=3k+1$.

3.3. **The $\mathbb{Z}$-grading.** We consider now the $\mathbb{Z}$-grading of $L=sl_2$, where $L_{-1}=\mathbb{K}e$, $L_0=\mathbb{K}h$, $L_1=\mathbb{K}f$ and $L_i=0$ for all $i\notin\{-1,0,1\}$.

We have the following result by Repin [24].

Proposition 3.7. *Let*

$$\chi_n(sl_2^{\mathbb{Z}}) = \sum_{\substack{\sigma\vdash p,\,\tau\vdash q\\ \pi\vdash r}} m_{\sigma,\tau,\pi}\chi_\sigma\otimes\chi_\tau\otimes\chi_\pi$$

be the (p,q,r)-th cocharacter of sl_2. Then, for every $\sigma \vdash p$, $\tau \vdash q$, $\pi \vdash r$, $m_{\sigma,\tau,\pi} \leq 1$. Moreover, $m_{\sigma,\tau,\pi} = 1$ if and only if $\sigma = (p)$, $\tau = (q)$, $\pi = (r)$ and the following conditions hold:

1. $p \neq n$, $q \neq n$, $r \neq n$;
2. $|p - r| \leq 1$.

As in the previous two cases, we get the asympthotics for the growth function of $\mathcal{L}_k^{\mathbb{Z}}(sl_2)$.

Proposition 3.8. *The growth function $g(n) = g(n)(\mathcal{L}_k^{\mathbb{Z}}(sl_2))$ of $\mathcal{L}_k^{\mathbb{Z}}(sl_2)$ is a polynomial in n of degree $3k-1$.*

Proof. By Proposition 3.7,

$$\chi_{p,q,r}(sl_2^{\mathbb{Z}}) = \sum \boxed{p} \otimes \boxed{q} \otimes \boxed{r}$$

where $m = p+q+r$, $p \neq m$, $q \neq m$, $r \neq m$ and $|p-r| \leq 1$.

For m even we have that

$$\begin{aligned}\chi_{p,q,r}(sl_2^{\mathbb{Z}}) = & \sum_{s_1=0}^{\lfloor \frac{m-1}{2} \rfloor} \boxed{\tfrac{m-2s_1}{2}} \otimes \boxed{2s_1} \otimes \boxed{\tfrac{m-2s_1}{2}} \\ & + \sum_{s_2=0}^{\frac{m-2}{2}} \boxed{\lfloor \tfrac{m-2s_2-1}{2} \rfloor + 1} \otimes \boxed{2s_2+1} \otimes \boxed{\lfloor \tfrac{m-2s_2-1}{2} \rfloor} \\ & + \sum_{s_2=0}^{\frac{m-2}{2}} \boxed{\lfloor \tfrac{m-2s_2-1}{2} \rfloor} \otimes \boxed{2s_2+1} \otimes \boxed{\lfloor \tfrac{m-2s_2-1}{2} \rfloor + 1}.\end{aligned}$$

Using arguments similar to those of Proposition 3.2 we obtain

$$\begin{aligned} a_m & \approx \sum_{s_1=0}^{\frac{m-1}{2}} s_T\left(\boxed{\tfrac{m-2s_1}{2}}\right) s_U\left(\boxed{2s_1}\right) s_V\left(\boxed{\tfrac{m-2s_1}{2}}\right) \\ & = \sum_{s_1=0}^{\frac{m-1}{2}} \binom{\frac{m-2s_1}{2}+k-1}{k-1}^2 \binom{2s_1+k-1}{k-1}. \end{aligned}$$

and consequently $g(n)$ is a polynomial in n of degree $3k-1$. □

Corollary 3.9. $\mathrm{GKdim}_k^{\mathbb{Z}}(sl_2) = 3k-1$.

4. The $\mathbb{Z}_n$-graded GK dimension of sl_n

In this section we compute the exact value of the $\mathbb{Z}_n$-graded GK dimension of sl_n.

Let us consider the relatively-free algebra of sl_n graded by $\mathbb{Z}_n$ with the grading of Vasilovsky (see [26] and [11]). Let $(sl_n)_i = \mathrm{span}_{\mathbb{K}}\{e_{pq} : |q-p| = i\}$, for all $i \in \{0, \dots, n-1\}$, where the e_{pq}'s are the usual matrix units and $|\cdot|$ is the reduction modulo n.

We shall construct $\mathcal{L}_k^{\mathbb{Z}_n}(sl_n)$ as a generic matrix algebra. For this purpose we consider $X = \{x_{ij}^{(r)} | i,j = 1, \dots, n-1, r = 1, \dots, k\}$ and for every $r = 1, \dots, k$, let us

consider the generic $n\times n$ matrices with entries from the algebra of the commutative polynomials $\mathbb{K}[X]$:

$$\begin{aligned} A_0^r &= \sum_{i=1}^{n-1} x_{ii}^{(r)} e_{ii} - \left(\sum_{i=1}^{n-1} x_{ii}^{(r)}\right) e_{nn}, \\ A_i^r &= \sum_{|q-p|=i} x_{pq}^{(r)} e_{pq}. \end{aligned}$$

We denote by $L_k(A)$ and $\mathbb{K}_k(A)$ respectively the Lie algebra and the associative unitary algebra generated by the A_i^r's. let us observe that $L_k(A)$ is contained in $\mathbb{K}_k(A)$. We have the following results (see [23]).

Theorem 4.1. *For every n, $k \in \mathbb{N}$, $k \geq 2$, we have $\mathcal{L}_k^{\mathbb{Z}_n}(sl_n) \cong L_k(A)$.*

In [3] the author proved the existence of a graded field of quotients Q for associative unitary G-prime algebras provided G to be either an abelian or an ordered group. The next result is easy to be proved.

Proposition 4.2. *For each $k, n \in \mathbb{N}$, $n \geq 2$ the algebra $\mathbb{K}_k(A)$ is a $\mathbb{Z}_n$-prime algebra.*

We consider Q to be the $\mathbb{Z}_n$-graded algebra of central quotients of $\mathbb{K}_k(A)$, always existing due to Proposition 4.2. We denote by Z the (graded)-center of Q which is a (graded) field. We observe that Z contains $\mathbb{K}$. Following word by word the computations by Procesi (see [22], Part II, Section 1), we have the next result.

Proposition 4.3. *Let $k, n \in \mathbb{N}$, where $k \geq 2$, then*

$$\mathrm{tr.deg}_{\mathbb{K}} Z = k(n^2-1) - n + 1.$$

Because $L_k(A)$ is contained in $\mathbb{K}_k(A)$, due to Proposition 4.3 and Theorem 5.7 of [10], we have the following.

Proposition 4.4. *Let $k, n \in \mathbb{N}$, where $k \geq 2$, then*

$$\mathrm{GKdim}_k^{sl_n} \leq k(n^2-1) - n + 1.$$

We observe that the growth function $g(r)$ of $L_k(A)$ depends on the number of linearly independent polynomials of degree r appearing in one of its entries, to say (a, b), hence from now on we are going to investigate such a number in such an entry. Let us set

$$X^* = \{x_{ij}^{(r)} | i, j = 1, \ldots, n-1, \ i \neq j, \ r = 1, \ldots, k\}.$$

We observe that for each polynomial m of any degree appearing in the entry (a, b) of $\mathbb{K}_k(A)$ having no variables appearing in one of the A_i^0 (so in the variables from X^*), we can construct a non-zero monomial M of the same degree of $L_k(A)$ having m as one of its summands in its (a, b)-entry. More precisely, if m is generated by the product $A_{i_1} \cdots A_{i_r}$ we consider $M = [A_{i_1}, \ldots, A_{i_r}]$.

Let $\sigma \in S_r$, then we consider the natural left action of GL_r on $A_{i_1} \cdots A_{i_r}$ such that $\sigma(A_{i_1} \cdots A_{i_r}) = A_{i_{\sigma_{-1}(1)}} \cdots A_{i_{\sigma^{-1}(r)}}$. We observe that $M = \sum_{\varphi \in \Phi} \sigma(A_{i_1} \cdots A_{i_r})$, where $|\Phi| = 2^{r-1}$ and each permutation of Φ is a product of cycles in which at most one cycle has length greater than 2. We have the next result.

Proposition 4.5. *In the previous notation and hipothesis, the left-normed Lie monomial $M = [A_{i_1}, \ldots, A_{i_r}]$ contains m in its (a, b)-entry.*

Proof. Straightforward computations show that the result is true for $r = 2$. Let $r > 2$, then suppose the proposition true for $r-1$. We observe that the (a, b) entry of M is $p_{ab'}x_{b'b} - x_{ab''}p_{b''b}$, where $p_{ab'}, p_{b''b}$ are entries of $[A_{i_1}, \dots, A_{i_{r-1}}]$. Due to the induction hipotheses, $p_{ab'} = m_{ab'} + p'$, where $m_{ab'}$ is the monomial appearing in the (a, b')-entry of $A_{i_1} \cdots A_{i_{r-1}}$ and p' is a polynomial. If the assertion is not true, there exists a monomial $x_{b''b'''} \cdots x_{a'b}$ of $p_{b''b}$ such that $m_{ab'}x_{b'b} - x_{ab''}x_{b''b'''} \cdots x_{a'b} = 0$. We note that at least the first and last variable of $m_{ab'}$ are different than $x_{b''b'''}$ and $x_{a'b}$. It turns out that there exist a submonomial $l := x_{ij} \cdots x_{a'b}$ of $x_{ab''}x_{b''b'''} \cdots x_{a'b}$ such that $x_{ij} \cdots x_{a'b}x_{ab''}x_{b''b'''} \cdots x_{a''j} = m_{ab'}x_{b'b}$ that means $a = b = i$. It also means that there exists a permutation, which is a product of two cycles, $\varphi \in \Phi$ such that $x_{ij} \cdots x_{a'b}x_{ab''}x_{b''b'''} \cdots x_{a''j}$ and $m_{ab'}x_{b'b}$ are the (a, a) entry of $\varphi(A_{i_1}, \dots, A_{i_{r-1}})$ and $A_{i_1}, \dots, A_{i_{r-1}}$ respectively, i.e., the latter are linearly dependent. Because each permutation of Φ is a product of cycles in which at most one cycle has length greater than 2, we have $\varphi(A_{i_1}, \dots, A_{i_{r-1}})$ and $A_{i_1}, \dots, A_{i_{r-1}}$ are linearly dependent if and only if $m_{ab'} = l$, that is (see [26]) if and only if A_{i_r} has $\mathbb{Z}_n$-degree 0. □

From now on we shall consider sl_n, $n \geq 3$. We observe that left-normed monomials with different multidegree are linearly independent. We consider the ordered set $S = \{m_1, \dots, m_s\}$ of linearly independent monomials of degree r appearing in the entry (a, b) of a product of generators of $\mathbb{K}_k(A)$ of $\mathbb{Z}_n$-degree different than 0. Due to the previous observation, we may consider the m_i's as generated by the non-zero monomials $M_1, \dots, M_s$ in $\mathbb{K}_k(A)$ with the same multidegree, i.e., if m_1 is generated by $M_1 := A_{i_1} \cdots A_{i_r}$, then for each $t \in \{2, \dots, s\}$ there exists $\sigma_t \in S_r$ such that m_t is generated by $\sigma_t(M_1) := M_t = A_{i_{\sigma_t^{-1}(1)}} \cdots A_{i_{\sigma_t^{-1}(r)}}$. We take now the set S' of non-zero left-normed monomials L_t of $L_k(A)$ as above corresponding to each t. We observe that each summand of L_t is a monomial of $\mathbb{K}_k(A)$ obtained by M_t applying a permutation $\varphi \in \Phi$.

Due to the fact that linearly independent monomials of $\mathbb{K}_k(A)$ grow polynomially whereas the monomial identities of a single monomial grow factorially, for sufficiently large r and for each M_i, M_j, it is possible to find permutations σ, τ such that $\sigma(M_i) = M_i$, $\tau(M_j) = M_j$ and $\varphi(\sigma(M_i)) \neq \varphi(\tau(M_j))$, $\varphi \in \Phi$. Hence we have the following result.

Proposition 4.6. *For sufficiently large r, S' is a linearly independent set.*

In an analogous way, we consider now polynomials m of any degree appearing in the entry (a, b) of $\mathbb{K}_k(A)$ in the variables from X. We observe that $\mathrm{GKdim}(\mathbb{K}_k(A)) = \mathrm{GKdim}(\mathcal{M})$, where $\mathcal{M}$ is the finitely generated $\mathbb{K}[X]$-module contained in $\mathbb{K}_k(A)$ formed by all products $A_{i_1} \cdots A_{i_r}$ such that $A_{i_1} \neq A_{i_2}$ and for each initial submonomial $A_{i_1} \cdots A_{i_l}$, $l \leq r$ such that $A_{i_1} \cdots A_{i_l}$ has $\mathbb{Z}_n$-degree 0 we have that $A_{i_{l+1}}$ has not $\mathbb{Z}_n$-degree 0. In light of this, we have that Proposition 4.6 gives us the next.

Proposition 4.7. *Let $k \in \mathbb{N}$, then* $\mathrm{GKdim}_k^{\mathbb{Z}_n}(sl_n) \geq \mathrm{GKdim}(\mathcal{M})$.

Combining Propositions 4.4 and 4.7 we have the following result.

Theorem 4.8. *Let $k \in \mathbb{N}$, then* $\mathrm{GKdim}_k^{\mathbb{Z}_n}(sl_n) = k(n^2 - 1) - n + 1$.

5. Conclusions

We want to draw out a parallelism with the arguments used by Procesi and the first of the authors in order to compute the $\mathbb{Z}_n$-graded Gelfand-Kirillov dimension of $M_n(\mathbb{K})$ (see [22] and [10]).

We introduce the notion of *G-prime (Lie) algebra*.

Definition 5.1. A G-graded ideal P of L is called *G-prime* if $aH \subseteq P$ for G-homogeneous element $a \in L$ and a G-graded ideal H of L, then either $a \in P$ or $H \subseteq P$. If (0) is a G-prime we say that L is a *G-prime algebra*.

When the grading group is not specified, we shall refer to *graded prime* algebras.

We have that the relatively-free algebras of sl_2 endowed with the gradings introduced in the previous section are graded prime.

Proposition 5.2. *The relatively-free G-graded algebra of sl_2 in k variables $\mathcal{L}_k^G(sl_2)$ is G-prime.*

Proof. We argue only in the case $G = \mathbb{Z}_2$ because the other cases may be treated analogously. Suppose that $aH = 0$ for a $\mathbb{Z}_2$-homogeneous element a of degree i and a $\mathbb{Z}_2$-graded ideal $H \neq 0$ of $\mathcal{L}_k^{\mathbb{Z}_2}(sl_2)$. Since $H = H_0 \oplus H_1$ is a non-trivial $\mathbb{Z}_2$-graded ideal then $H_0 \neq 0$ and $H_1 \neq 0$ where H_0 and H_1 are the homogeneous component of H of the degree 0 and 1 respectively. By simple calculations $aH_j = 0$ with $j \neq i$ implies that $a = 0$. □

We have already pointed out that the associative algebras $\mathbb{K}_k(A)$ generated by the generic homogeneous elements of the $\mathbb{Z}_n$-graded relatively-free algebra of sl_n are $\mathbb{Z}_n$-prime (Proposition 4.2). Of course, due to the theory of Balaba [3], we have the existence of a (graded) field of quotients whose transcendence degree over the ground field equals the GK dimension of $\mathbb{K}_k(A)$. The fact that the relatively-free algebras $L_k(A)$ of sl_n is prime in a Lie sense (Proposition 5.2) gives us a hope that what happened in the associative case is maybe true in the Lie case, i.e. there exists a graded central localization of $L_k(A)$ such that the "generalized transcendence degree" of its "center" over the ground field gives a measure of the graded GK dimension of $L_k(A)$. We recall that a definition of central localization for Lie algebras may be found in [20].

For the sake of completeness we want to show a consequence of some results of Procesi and Razmyslov (see [21] and [23]) in the ordinary case. In particular, let $C_{kn}^{(0)}$ be the trace algebra of $\mathbb{K}_k(A)$, the associative unitary algebra generated by k ungraded generic traceless $n \times n$ matrices over the field $\mathbb{K}$, and $T_{kn}^{(0)}$ the mixed trace algebra of $\mathbb{K}_k$. Then we have the next result.

Theorem 5.3. *For $k, n \geq 2$ we have*

$$\mathrm{GKdim}(T_{kn}^{(0)}) = \mathrm{GKdim}(C_{kn}^{(0)}) = (k^2 - 1)(n - 1).$$

Due to the fact that $\mathbb{K}_k(A)$ is a prime algebra, we have the ungraded analog of Proposition 4.3.

Proposition 5.4. *Let $k, n \in \mathbb{N}$, where $k \geq 2$, then*

$$\mathrm{GKdim}(\mathbb{K}_k(A)) = (k^2 - 1)(n - 1).$$

In light of Theorem 4.8 and Proposition 5.4, it is reasonable to consider a graded generalization of Theorem 5.3.

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